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A GENERAL THEORY OF UNIQUENESS AND STABILITY IN ELASTIC–PLASTIC SOLIDS

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SUMMARY

A SUFFICIENT condition is established for uniqueness of the boundary-value problem set by given velocities on a part of the surface of a body and given nominal traction-rates on the remainder. No restriction is placed on changes in geometry either in the boundary-value problem itself or in the postulated material properties, which are a generalization of those conventionally assumed for workhardening elastic–plastic solids. The solution, when it is unique, is characterized by an extremum principle.

Stability under dead loading is also examined, and a criterion is proposed. This has a formal resemblance to the uniqueness criterion, but differs from it in a significant respect when the body is partly plastic, so that stability and uniqueness need not be parallel properties.

Finally, the present theorems are compared with those previously proved for rigid-plastic solids (HILL 1957a, b, d).

1. SCOPE OF PAPER

THE theory of partly plastic solids is especially concerned with the following boundary-value problem. At a generic stage in a process of quasi-static distortion the current shape of the body and the internal distribution of stress are supposed to have been determined already, together with the existing state of hardening and mechanical properties generally. The incremental changes in all these variables have now to be calculated for a further infinitesimal variation of the surface loads and geometrical constraints. When the total strain remains small, and the positional changes and rotations of material elements are neglected, the boundary-value problem always has a unique solution when the workhardening is monotonic and the yield function and plastic potential are identical (MELAN 1938; a somewhat more rigorous and general proof can be found in HILL 1950, pp. 55-6).

However, the really typical plastic problems involve changes in geometry that cannot be disregarded, and here the question of uniqueness has remained open until very recently, when an answer was given for rigid–plastic solids (HILL 1957a). Broadly speaking, it is found that a unique solution is certain only when the rate of hardening exceeds a determinable critical value, depending on the particular situation. In the present paper a similar result is proved for an extensive class of elastic–plastic solids, including metals, with mechanical properties wider in important respects than those usually contemplated. Moreover, the solution, when it is unique, is characterized by an extremum principle, analogous to that for rigid–plastic solids (HILL 1957b).

In elastic solids the problem of instability is closely related to that of uniqueness (HILL 1957c); in plastic solids the connexion is looser, but still significant. For this reason we also investigate the question of stability here, and treat it from the most general standpoint, namely via the energy criterion. Previous work is unsatisfactory in that either the mechanical properties, the field equations, or the boundary conditions, are often not properly formulated for a situation where rotations of material elements are large compared with the strains. Furthermore, the occurrence of instability has invariably been taken as synonymous with the existence of infinitesimally near positions of equilibrium; this may be quite unjustified when the system is non-linear or non-conservative. For these reasons alone it is hardly surprising that theoretical predictions of the onset of instability should often have disagreed markedly with each other and with experiment. It is not, however, the present intention to review a confused literature nor to attempt any correlation with experiment but to make a fresh start and establish a broad basic theory free at least from the objections mentioned.

2. Basis for Analysis of Stress

The typical traction boundary-condition considered here is that the change in the *load* vector on an infinitesimal surface element is assigned, irrespective of changes in its area and orientation. Thus, at any stage t during a quasi-static process of deformation there is given, not the rate of change of the 'true' traction, but the rate of change of the 'nominal' traction based on the configuration at that instant. To formulate this boundary condition we introduce the unsymmetrical 'nominal stress' tensor $s_{ij}(t)$ associated with certain fixed rectangular axes. At any stage $t + \delta t$ subsequent to t, $(s_{ij} + \delta s_{ij})\, dS$ is the jth component of the force on the plane element which at t was perpendicular to the ith axis and of infinitesimal area dS. Correspondingly, the force at $t + \delta t$ on a plane area whose vectorial area was $l_i\, dS$ at t has components $(F_j + \delta F_j)\, dS$ where

$$F_j = l_i\, s_{ij}, \qquad \delta F_j = l_i\, \delta s_{ij}.$$

F_j is the 'nominal traction' based on the configuration at t. The boundary condition is then that

$$\dot{F}_j = l_i\, \dot{s}_{ij} \tag{1}$$

is prescribed.

In equilibrium and in the absence of body forces,

$$\frac{\partial \dot{s}_{ij}}{\partial x_i} = 0 \tag{2}$$

where x_i marks the position of elements at stage t. The angular equations of equilibrium are satisfied through the association of s_{ij} with the symmetric 'true stress' tensor σ_{ij} referred to the same axes. If $\dot{\sigma}_{ij}^{\circ}$ denotes its rate of change following the same element,

$$\dot{s}_{ij} - \dot{\sigma}_{ij}^{\circ} = \sigma_{ij} \frac{\partial v_k}{\partial x_k} - \sigma_{jk} \frac{\partial v_i}{\partial x_k}$$

where v_i is the velocity. This is an immediate specialization of Murnaghan's well-known formula connecting the true and nominal stress tensors.

Now the material properties found by experiment are given, in the first instance, in terms of the true stress tensor σ_{ij} associated with rectangular axes rotating with the element and so having spin $\frac{1}{2}$ curl v. From the transformation rule for tensors it is easily shown that

$$\dot{\sigma}_{ij} - \dot{\sigma}_{ij}^{\circ} = \sigma_{ik}\,\alpha_{kj} + \sigma_{jk}\,\alpha_{ki}$$

where

$$\alpha_{ij} = \tfrac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$$

is the antisymmetric part of the velocity gradient tensor. (It is worth noting that $\dot{\sigma}_{ij}$ and $\dot{\sigma}_{ij}^{\circ}$ have the same scalar product with any tensor whose principal axes coincide with those of σ_{ij}.) On combining the last two equations :

$$\dot{s}_{ij} - \left(\dot{\sigma}_{ij} + \sigma_{ij}\frac{\partial v_k}{\partial x_k}\right) = \sigma_{ik}\frac{\partial v_j}{\partial x_k} - (\sigma_{ik}\,\epsilon_{jk} + \sigma_{jk}\,\epsilon_{ik}). \tag{3}$$

3. MATERIAL PROPERTIES

In an element whose response to any further infinitesimal change of stress is purely elastic (i.e. an element that has either never yielded or is now partly unloaded after being plastic), the isothermal relations between stress-rate and strain-rate are taken to be homogeneous and linear, and are written as

$$\dot{\sigma}_{ij} + \sigma_{ij}\frac{\partial v_k}{\partial x_k} = g_{ijkl}\,\epsilon_{kl} \tag{4}$$

where the moduli g_{ijkl} depend on the current stress and orientation of the axes of anisotropy, and possibly also on the entire strain-history if the element has previously been plastic. Moreover, and this is of paramount importance for what follows, it is stipulated that

$$g_{ijkl} = g_{klij} \tag{5}$$

in addition to the necessary symmetry with respect to interchange of i and j, and the arbitrarily arranged symmetry with regard to interchange of k and l. Equation (4), with (5), is certainly justifiable when the element behaves as a perfectly elastic solid during *complete* unloading from the current stress : it follows directly from a formula of Cauchy's in the theory of finite elastic strain*.

To simplify the notation (4) is written as

$$\dot{\boldsymbol{\sigma}} + \theta\boldsymbol{\sigma} = \boldsymbol{K}\boldsymbol{\epsilon} \tag{6}$$

where tensor elements are regarded as constituting vectors in a 9-space. $\boldsymbol{K}$ is a symmetric 9×9 matrix and θ is the fractional rate of volume increase. If the total strain recoverable by complete unloading is infinitesimal, $\boldsymbol{K}$ would be positive-definite for any actual solid, according to the familiar thermodynamic argument in classical elasticity. On the other hand, $\boldsymbol{K}$ need not be positive-definite when the

*See, for example, equation (18) of HILL (1957c); note that there the stress-rate symbol refers to components associated with fixed axes and differs from the present one.

recoverable strain is finite, and indeed there is no need so to restrict K in the subsequent analysis*.

For a plastic element the simplest hypothesis is

$$\dot{\sigma} + \theta\sigma = K(\epsilon - \epsilon_p) \tag{7}$$

where ϵ_p is the plastic part of the strain-rate, and K is a symmetric matrix depending only on strain-history. Suppose that

$$\epsilon_p = \begin{cases} h^{-1}[\mathbf{n}(\dot{\sigma} + \theta\sigma)]\,\mathbf{n} & \text{when} \quad \mathbf{n}(\dot{\sigma} + \theta\sigma) \geqslant 0 \\ 0 & \ldots\ldots\ldots\ldots\ldots \leqslant 0 \end{cases} \tag{8}$$

where h is a positive scalar measure of the current rate of workhardening and depends on σ as well as the strain-history. The ' direction ' of the plastic strain-rate is specified by the unit vector $\mathbf{n}$, which also depends on σ and strain-history; $\mathbf{n}$ is stipulated to be deviatoric so that there is no permanent volume change. Since θ is directly expressible in terms of stress-rate through the elastic moduli, the equation

$$\mathbf{n}(\dot{\sigma} + \theta\sigma) = 0$$

effectively defines those stress-rate vectors that are tangential to the yield surface at the current stress point, and thereby the surface element. This does not have $\mathbf{n}$ as its normal and so yielding is affected by the hydrostatic part of the true stress. For present purposes no restrictions need be placed on the shape of the surface, and in particular we do not make the customary assumption of convexity.

Note that, in view of the definition of $\dot{\sigma}$, equation (8) conforms to the minimum requirement that it is satisfied identically by a rigid-body spin of the element together with the stress state.

To invert the relations (7), with (8), we first remark that

$$\mathbf{n}K\epsilon = \mathbf{n}(\dot{\sigma} + \theta\sigma).\begin{cases} (1 + h^{-1}\,\mathbf{n}K\mathbf{n}) & \text{when} \quad \mathbf{n}(\dot{\sigma} + \theta\sigma) \geqslant 0 \\ 1 & \ldots\ldots\ldots\ldots\ldots \leqslant 0 \end{cases}$$

provided $1 + h^{-1}\,\mathbf{n}K\mathbf{n} > 0$. This inequality we satisfy *a fortiori* for any $h > 0$ by requiring

$$\mathbf{n}K\mathbf{n} > 0 \text{ in a plastic element.} \tag{9}$$

That is, the K quadratic form is positive for *deviatoric* vectors; this has the interpretation that a strain-rate in direction $\mathbf{n}$ constitutes loading. Then, in a plastic element,

$$\dot{\sigma} + \theta\sigma = K\epsilon - \begin{cases} \dfrac{\mathbf{n}K\epsilon}{(\mathbf{n}K\mathbf{n} + h)}K\mathbf{n} & \text{when} \quad \mathbf{n}K\epsilon \geqslant 0 \\ 0 & \ldots\ldots\ldots\ldots \leqslant 0. \end{cases} \tag{10}$$

If the rate of hardening is zero (8) is replaced by

$$\epsilon_p = \begin{cases} \gamma\mathbf{n}, & \text{where} \quad \gamma \geqslant 0, \quad \text{when} \quad \mathbf{n}(\dot{\sigma} + \theta\sigma) = 0 \\ 0 & \ldots\ldots\ldots\ldots\ldots < 0 \end{cases} \tag{8'}$$

*It is of more than academic interest that the eventual uniqueness theorem and extremum principle then apply also to some ' unreal ' materials. For it can happen that a boundary-value problem is solvable in the first instance more easily for some special kind of unreal material; this solution can then be of use in finding that for the real material.

while (10) still holds, with $h = 0$.

Now in fact the only elastic-plastic solids that have been studied experimentally are the common metals. These have elastic moduli so great compared with any stress supportable by the material that there is perhaps no point in retaining the $\theta\sigma$ term nor in postulating any particular stress-dependence of K itself in the elastic range, though we may still envisage measurable changes in K due to *plastic* straining, especially through changes in the state of anisotropy. Equations (7), (8) and (10) then reduce to the generally accepted relations*, with $\mathbf{n}$ as the unit normal to the yield surface (now independent of hydrostatic stress).

4. Fundamental Identities and Inequalities

We shall later need to refer to the quantity

$$E(\mathbf{v}) = \tfrac{1}{2}(\dot{\sigma} + \theta\sigma)\,\epsilon - \tfrac{1}{2}\sigma_{ik}\left(2\epsilon_{ij}\,\epsilon_{jk} - \frac{\partial v_j}{\partial x_i}\frac{\partial v_j}{\partial x_k}\right) \tag{11}$$

which is a function of velocity gradient, equal to $\tfrac{1}{2}\dot{s}_{ij}\,\partial v_j/\partial x_i$ by (3). Moreover, in view of (6) and (10),

$$\dot{s}_{ij}\,d\left(\frac{\partial v_j}{\partial x_i}\right) = dE, \qquad \dot{s}_{ij} = \frac{\partial E}{\partial\,(\partial v_j/\partial x_i)}.$$

That the left-hand differential is perfect could also be recognised in advance from the symmetry, with respect to interchange of ij and kl, of the matrix of coefficients in the relations between $\dot{s}_{ij}$ and $\partial v_i/\partial x_i$.

Now let ϵ^*, ϵ denote distinct strain-rates and $\dot{\sigma}^*$, $\dot{\sigma}$ the corresponding stress-rates in an element in a given condition and state of stress. We write $\Delta\epsilon = \epsilon^* - \epsilon$ and similarly for the differences of other starred and unstarred quantities. Note that when the element is plastic $\Delta\dot{\sigma}$ and $\Delta\epsilon$, regarded as single vectors, do not necessarily correspond in the sense of being related by (10).

Lemma I. In an elastic element there is the immediate identity

$$\Delta(\dot{\sigma} + \theta\sigma)\,\Delta\epsilon = \Delta\epsilon\,K\,\Delta\epsilon. \tag{12}$$

In a plastic element, for $h \geqslant 0$,

$$\Delta(\dot{\sigma} + \theta\sigma)\,\Delta\epsilon \geqslant \Delta\epsilon\,K\,\Delta\epsilon - \frac{(\mathbf{n}K\,\Delta\epsilon)^2}{\mathbf{n}K\mathbf{n} + h}. \tag{13}$$

This lemma is used in deriving the uniqueness criterion (Section 5).

Proof. The equality in (13) holds when both strain-rates call for additional loading, from (11) and the symmetry of K. The inequality holds when $\mathbf{n}\,K\,\epsilon^* \geqslant 0$ and $\mathbf{n}K\epsilon < 0$. For then the left side of (13) is

$$\Delta\epsilon\,K\Delta\epsilon - \frac{(\mathbf{n}K\epsilon^*)(\mathbf{n}K\Delta\epsilon)}{\mathbf{n}K\mathbf{n} + h}$$

and this exceeds the right side. Finally, if both strain-rates produce unloading, their difference satisfies (12) and hence (13) *a fortiori* (with equality only when $\mathbf{n}K\,\Delta\epsilon = 0$).

*Though the form of (10), and the modification of the loading criterion through the inclusion of elastic anisotropy, is perhaps a new generalization.

Lemma II. In an elastic element there is the easily verified identity

$$2\,(\dot{\boldsymbol{\sigma}} + \theta\boldsymbol{\sigma})\,\Delta\boldsymbol{\epsilon} + \Delta\boldsymbol{\epsilon}\,\boldsymbol{K}\,\Delta\boldsymbol{\epsilon} = \Delta\,[(\dot{\boldsymbol{\sigma}} + \theta\boldsymbol{\sigma})\,\boldsymbol{\epsilon}]. \tag{14}$$

In a plastic element, for $h \geqslant 0$,

$$2\,(\dot{\boldsymbol{\sigma}} + \theta\boldsymbol{\sigma})\,\Delta\boldsymbol{\epsilon} + \left[\Delta\boldsymbol{\epsilon}\,\boldsymbol{K}\,\Delta\boldsymbol{\epsilon} - \frac{(\mathbf{n}\boldsymbol{K}\,\Delta\boldsymbol{\epsilon})^2}{\mathbf{n}\boldsymbol{K}\mathbf{n} + h}\right] \leqslant \Delta\,[(\dot{\boldsymbol{\sigma}} + \theta\boldsymbol{\sigma})\,\boldsymbol{\epsilon}]. \tag{15}$$

This lemma is used in obtaining the extremum principle (Section 6). A weaker inequality sufficed in the existing proof of this principle when geometry changes are neglected (K being positive-definite) : in that event the square bracket, which is positive by Cauchy's inequality, is omitted from the left side of (15).

Proof. When $\mathbf{n}\boldsymbol{K}\boldsymbol{\epsilon} \geqslant 0$ the left side of (15) is equal to

$$\Delta\left[\boldsymbol{\epsilon}\boldsymbol{K}\boldsymbol{\epsilon} - \frac{(\mathbf{n}\boldsymbol{K}\boldsymbol{\epsilon})^2}{\mathbf{n}\boldsymbol{K}\mathbf{n} + h}\right]$$

by (10) and the symmetry of K. If $\mathbf{n}\,\boldsymbol{K}\,\boldsymbol{\epsilon}^* \geqslant 0$ this is just the right side of (15) which is therefore an equality. If, however, $\mathbf{n}\,\boldsymbol{K}\,\boldsymbol{\epsilon}^* < 0$ the right side exceeds the left by $(\mathbf{n}\boldsymbol{K}\boldsymbol{\epsilon}^*)^2/(\mathbf{n}\boldsymbol{K}\mathbf{n} + h)$.

When $\mathbf{n}\boldsymbol{K}\boldsymbol{\epsilon} < 0$ the left side of (15) is equal to

$$\Delta\,(\boldsymbol{\epsilon}\boldsymbol{K}\boldsymbol{\epsilon}) - \frac{(\mathbf{n}\boldsymbol{K}\,\Delta\boldsymbol{\epsilon})^2}{\mathbf{n}\boldsymbol{K}\mathbf{n} + h}.$$

If $\mathbf{n}\boldsymbol{K}\boldsymbol{\epsilon}^* \geqslant 0$ the right side of (15) is

$$\Delta\,(\boldsymbol{\epsilon}\boldsymbol{K}\boldsymbol{\epsilon}) - \frac{(\mathbf{n}\boldsymbol{K}\boldsymbol{\epsilon}^*)^2}{\mathbf{n}\boldsymbol{K}\mathbf{n} + h}$$

which exceeds the left since $(\mathbf{n}\,\boldsymbol{K}\,\Delta\,\boldsymbol{\epsilon})^2 > (\mathbf{n}\,\boldsymbol{K}\,\boldsymbol{\epsilon}^*)^2$. If $\mathbf{n}\,\boldsymbol{K}\,\boldsymbol{\epsilon} < 0$ the right side of (15) is just $\Delta\,(\boldsymbol{\epsilon}\,\boldsymbol{K}\,\boldsymbol{\epsilon})$ and is not less than the left (being equal when $\mathbf{n}\,\boldsymbol{K}\,\Delta\,\boldsymbol{\epsilon} = 0$).

5. Uniqueness Criterion

We regard the current distribution of stress in a body as given, together with the material properties at every point. For simplicity body forces are omitted since their mode of inclusion is sufficiently obvious. The nominal traction-rate $\dot{\mathbf{F}}$ is specified on a part S_F of the current surface and the velocity $\mathbf{v}$ on the remainder S_v. These conditions and the field equations (2), (3), (6) and (10) set a boundary-value problem for the internal velocity field.

Suppose that there could be two distinct solutions (not differing merely by a rigid-body motion when $S_v = 0$) and denote their difference by $\Delta\,\mathbf{v}$. Then, from (1) and (2), with S and V as the present surface and volume,

$$0 = \int \Delta\dot{\mathbf{F}}\,\Delta\mathbf{v}\,dS = \int \Delta\dot{s}_{ij}\,\Delta\left(\frac{\partial v_j}{\partial x_i}\right)dV$$

$$= \int \Delta\,(\dot{\boldsymbol{\sigma}} + \theta\boldsymbol{\sigma})\,\Delta\boldsymbol{\epsilon}\,dV - \Sigma\,(\Delta\mathbf{v})$$

where

$$\Sigma\,(\mathbf{v}) = \int \sigma_{ik}\left(2\epsilon_{ij}\,\epsilon_{jk} - \frac{\partial v_j}{\partial x_i}\frac{\partial v_j}{\partial x_k}\right)dV.$$

A sufficient condition for uniqueness is therefore that

$$\int \Delta(\dot{\sigma} + \theta\sigma)\,\Delta\epsilon\,dV - \Sigma(\Delta\mathbf{v}) > 0$$

for the differences $\Delta\mathbf{v}$ of every distinct pair of continuous velocity fields taking the given values on S_v.

For applications it is preferable to replace this condition by a slightly weaker (over-sufficient) criterion which is a functional only of the single vector $\Delta\mathbf{v}$. From lemma I, when $h \geqslant 0$,

$$\int \Delta(\dot{\sigma} + \theta\sigma)\,\Delta\epsilon\,dV \geqslant H(\Delta\mathbf{v})$$

where

$$H(\mathbf{v}) = \int \epsilon K \epsilon\,dV - \int \frac{(\mathbf{n}K\epsilon)^2}{\mathbf{n}K\mathbf{n} + h}\,dV_p,$$

V_p being the *whole* of the plastically stressed zone. We thereby obtain the sufficient condition

$$H(\mathbf{w}) - \Sigma(\mathbf{w}) > 0 \tag{16}$$

for all continuous velocity fields $\mathbf{w}$ vanishing on S_v but not identically zero in V.

That the condition may well be often *over-sufficient* (i.e. not necessary) can be surmised from its being entirely independent of the particular boundary values of velocity and traction-rate.

H_0 will henceforward be written for H when $h = 0$ in V_p, corresponding to a non-hardening solid. Similarly H_∞ will be written for $\int \epsilon K \epsilon\,dV$, the value of H when $h \to \infty$ in V_p, corresponding to a solid which has not previously been plastically strained. Note that $H_0 \leqslant H \leqslant H_\infty$.

The following situations may be distinguished :

(i) $\Sigma < H_\infty$ for all $\mathbf{w}$ fields. The problem is unique for purely elastic behaviour, as has been shown elsewhere (HILL 1957c).

(ii) $\Sigma < H_0$ for all $\mathbf{w}$ fields. The problem is unique for all $h \geqslant 0$, since $H \geqslant H_0$ and $\Sigma < H$ always.

(iii) $\Sigma \leqslant H_0$ for all $\mathbf{w}$ fields, with equality for at least one. The problem is unique whenever $h > 0$ (workhardening solid) since $\Sigma \leqslant H_0 < H$. But nothing can be concluded when $h = 0$ everywhere in V_p (non-hardening solid) since $\Sigma = H_0 = H$ for at least one field.

(iv) $\Sigma > H_0$ for at least one field and $\Sigma < H_\infty$ for all. The problem is unique with certainty only for sufficiently great rates of hardening.

To clarify and give precision to (iv) consider an elastically-isotropic solid, for which $\mathbf{n}K\mathbf{n} = 2\mu$ where μ is the rigidity modulus. Then

$$H_\infty(\mathbf{w}) = \int (\lambda\phi^2 + 2\mu\eta^2)\,dV, \qquad H_0(\mathbf{w}) = H_\infty(\mathbf{w}) - \int 2\mu(\mathbf{n}\eta)^2\,dV_p, \tag{17}$$

where λ is the other Lamé constant, and η is the strain-rate and ϕ the fractional rate of volume increase corresponding to $\mathbf{w}$. If h/μ has the same value throughout

V_p, the uniqueness condition can be rearranged to give explicitly

$$\frac{h}{2\mu} > \beta \qquad \text{where} \qquad \beta = \text{Max}\left(\frac{\Sigma - H_0}{H_\infty - \Sigma}\right) \text{ in class } \mathbf{w}, \tag{18}$$

the *algebraic* maximum being implied.

6. Extremum Principle

When the criterion (16) is satisfied, the unique solution is characterized by the following extremum property. In the class of continuous velocity fields taking the given values on S_v the functional

$$\int E(\mathbf{v})\, dV - \int \dot{\mathbf{F}}\mathbf{v}\, dS_F \tag{19}$$

has an absolute minimum when $\mathbf{v}$ is the actual solution, E being as in (11). The least value is $\frac{1}{2}\left\{\int \dot{\mathbf{F}}\mathbf{v}\, dS_v - \int \dot{\mathbf{F}}\mathbf{v}\, dS_F\right\}$ from (3) and the usual transformation.

With $\Delta\mathbf{v} = \mathbf{v}^* - \mathbf{v}$, where $\mathbf{v}^*$ is any distinct field of the class,

$$\Delta \int \dot{\mathbf{F}}\mathbf{v}\, dS_F = \int \dot{\mathbf{F}}\Delta\mathbf{v}\, dS,$$

$\dot{\mathbf{F}}$ being given in the left-hand integral. From (3) this becomes

$$\int (\dot{\sigma} + \theta\sigma)\, \Delta\epsilon\, dV - \int \sigma_{ik}\left[2\epsilon_{jk}\, \Delta\epsilon_{ij} - \frac{\partial v_j}{\partial x_k}\, \Delta\left(\frac{\partial v_j}{\partial x_i}\right)\right] dV.$$

Now by (14) and (15)

$$2\int (\dot{\sigma} + \theta\sigma)\, \Delta\epsilon\, dV + H(\Delta\mathbf{v}) \leqslant \Delta \int (\dot{\sigma} + \theta\sigma)\, \epsilon\, dV.$$

There is also the identity

$$2\int \sigma_{ik}\left[2\epsilon_{jk}\, \Delta\epsilon_{ij} - \frac{\partial v_j}{\partial x_k}\, \Delta\left(\frac{\partial v_j}{\partial x_i}\right)\right] dV + \Sigma(\Delta\mathbf{v}) = \Delta\, \Sigma(\mathbf{v}).$$

Combining the last three equations and using (16), which holds for $\mathbf{w} = \Delta\mathbf{v}$ by hypothesis,

$$\Delta\left[\int (\dot{\sigma} + \theta\sigma)\, \epsilon\, dV - \Sigma(\mathbf{v}) - 2\int \dot{\mathbf{F}}\mathbf{v}\, dS_F\right] > 0.$$

This proves the extremum principle. When geometry changes are negligible (Σ and θ terms discarded, true and nominal stress-rates not distinguished) it reduces to a known result (Hill 1950, pp. 65-6).

The minimum is also analytic, as well as absolute, in the sense that, for any infinitesimal variation $\delta\mathbf{v}$, the first variation of (19) vanishes when $\mathbf{v}$ is the actual solution. For the inequality rests on the second-order term $H(\delta\mathbf{v}) - \Sigma(\delta\mathbf{v})$ and (by inspection of the proof of lemma II) on the integral of $(\mathbf{n}K\,\delta\epsilon)^2/(\mathbf{n}K\mathbf{n} + h)$ over that part of the plastic zone where neither of the actual and varied fields produce loading.

For the same reasons, even when the solution is not unique and (16) is not

satisfied, the variation principle exists in respect of each solution, though the extremum principle may not.

It is essential to keep in mind the distinction between the functionals in (16) and (19). In the elastic zone and in the part of the plastic zone where $\mathbf{n}\,K\,\epsilon \leqslant 0$, $(\dot{\sigma} + \theta\,\sigma)\,\epsilon$ is equal to $\epsilon\,K\,\epsilon$. Only in the loading part of the plastic zone, where $\mathbf{n}\,K\,\epsilon > 0$, is it equal to

$$\epsilon K \epsilon - \frac{(\mathbf{n}K\epsilon)^2}{\mathbf{n}K\mathbf{n} + h}.$$

By contrast, in the uniqueness functional the term containing h has to be included *everywhere* in the plastic zone, irrespective of the sign of $\mathbf{n}K\eta$.

Further insight into the connexion between the extremum principle and uniqueness criterion can be gained by studying the matter for a still more general class of solids. We describe the basic idea here only in outline. Let the (non-linear) relations between nominal stress-rate and velocity gradient be

$$\dot{s}_{ij} = \frac{\partial E}{\partial\,(\partial v_j/\partial x_i)}$$

where E is a function of the velocity gradient, depending also on strain history, with continuous first derivatives and sectionally continuous second derivatives. Supposing there could be two distinct solutions to the boundary-value problem,

$$\int \Delta \dot{s}_{ij}\,\Delta\left(\frac{\partial v_j}{\partial x_i}\right) dV = 0.$$

A sufficient condition for uniqueness is therefore that

$$\int \Delta\left\{\frac{\partial E}{\partial\,(\partial v_j/\partial x_i)}\right\} \Delta\left(\frac{\partial v_j}{\partial x_i}\right) dV > 0 \tag{20}$$

for all pairs of continuous velocity fields taking the prescribed values on S_v. By specializing (20) for pairs differing only infinitesimally, we see that it implies

$$\int c_{ijkl}\,\xi_{ij}\,\xi_{kl}\,dV > 0 \tag{21}$$

for arbitrary ξ_{ij} and for

$$c_{ijkl} = \frac{\partial^2 E}{\partial\,(\partial v_j/\partial x_i)\,\partial\,(\partial v_l/\partial x_k)}$$

corresponding to *any* velocity field.

Now, by the mean value theorem of the second order, when c_{ijkl} is a continuous function of velocity gradient,

$$\int\left[\Delta E - \frac{\partial E}{\partial\,(\partial v_j/\partial x_i)}\,\Delta\left(\frac{\partial v_j}{\partial x_i}\right)\right] dV = \tfrac{1}{2}\int \bar{c}_{ijkl}\,\Delta\left(\frac{\partial v_j}{\partial x_i}\right)\Delta\left(\frac{\partial v_l}{\partial x_k}\right) dV \tag{22}$$

where $\Delta\,\mathbf{v}$ is now the difference of *any* two fields, and the bar denotes a value for some velocity gradient on the 'join' of those for the two fields. From (21) and (23), with $\xi_{ij} = \Delta\,(\partial v_j/\partial x_i)$,

$$\int\left[\Delta E - \frac{\partial E}{\partial\,(\partial v_j/\partial x_i)}\,\Delta\left(\frac{\partial v_j}{\partial x_i}\right)\right] dV > 0.$$

(When c_{ijkl} is discontinuous an extension of the conventional mean-value theorem has to be invoked, and a corresponding interpretation placed on (21)). If, now, $\mathbf{v}$ is the actual unique solution a transformation of the last equation gives the extremum principle

$$\Delta\left[\int E dV - \int \dot{\mathbf{F}}\mathbf{v}\, dS_F\right] > 0, \tag{23}$$

which has thereby been derived as a consequence of (20).

This is an extension of the systematic procedure of HILL (1956a) for ' convex ' functions E, in that (20) can be satisfied without the integrand itself being always positive. This extension was given in essence by HILL (1957c, pp. 233-4) in connexion with rubber-like solids for which E is a quadratic form whose second derivatives are the elastic moduli. For elastic–plastic solids E is given by (11) in conjunction with (6) and (10); its second derivatives are discontinuous for velocity gradients such that $\mathbf{n} K \boldsymbol{\epsilon} = 0$.

7. STABILITY CRITERION

Suppose that a part of the surface of the body is rigidly constrained and that constant nominal tractions (dead loading) are maintained on the remainder throughout any movement from the position of equilibrium at stage t. Let $\mathbf{u}$ be an arbitrary virtual displacement, during the course of which the nominal stress changes from s_{ij} to $s_{ij} + \delta s_{ij}$ (still referred to stage t). In a further infinitesimal displacement $d\mathbf{u}$ the increase in internal energy exceeds the work of the external forces by the amount

$$\int\left\{(s_{ij} + \delta s_{ij})\, d\left(\frac{\partial u_j}{\partial x_i}\right)\right\} dV - \int (\mathbf{F}\, d\mathbf{u})\, dS = \int\left\{\delta s_{ij}\, d\left(\frac{\partial u_j}{\partial x_i}\right)\right\} dV,$$

where S, V and x_i are the surface, volume and position at stage t. The total excess in reaching the position $\mathbf{u}$ is the integral of this over the whole ' path.' A sufficient condition for stability at stage t is therefore that

$$\int\left\{\int \delta s_{ij}\, d\left(\frac{\partial u_j}{\partial x_i}\right)\right\} dV > 0, \tag{24}$$

where the inner integral is taken along any path leading to each geometrically possible configuration infinitesimally near the initial one. These configurations need not of course be positions of equilibrium since they are envisaged as being attained (if at all) by free motion following a transitory dynamical disturbance. However, it may happen that there is a certain path that can be traversed quasi-statically under the dead loads; in this event the integral vanishes, as may be shown by direct transformation or equivalently from the virtual work principle. Of course the integral can vanish in other circumstances also.

In a plastic solid the change in stress is not a single-valued function of the total displacement gradient, and so (and if only for this reason) the integral in (24) is strongly path-dependent. Among the paths leading to any one final position the most critical for stability is that which calls for the smallest increase in internal energy in the body as a whole. In attempting to determine this it seems desirable, at the outset, to approach the problem by calculating the least increase in energy

in bringing each element of the body *separately* to its final state of distortion; since this would generally violate the continuity of the material, the actual least value may be underestimated and the resulting stability criterion be over-sufficient. However, we shall later give a reason why this is not the case here.

In an element after a finite distortion there is a unique triad of mutually perpendicular directions which were also perpendicular initially, though not necessarily so during the straining. These are the 'principal directions' in the element. Let λ_r ($r = 1, 2, 3$) be the natural logarithms of the ratios of the final and initial lengths in these directions. The final distortion could be reached along a path at every stage of which the *true* strain-rate (based on the configuration at that stage) has principal axes coinciding with the principal directions and principal components proportional to the λ_r's. It seems very probable that this is actually the optimum path (at all events to *neighbouring* states), but a rigorous proof has not been found. We adopt it here as a hypothesis.

The calculation need only be correct to second order in the additional displacements, which are now taken to be infinitesimal. Let the true stress initially have normal (*not* principal) components σ_r along the strain-rate axes and suppose they finally become $\sigma_r + \delta\sigma_r$. At any intermediate stage on the optimum path the changes in the normal components are in proportion to the respective $\delta\sigma_r$'s, to first order, since the strain-rate components are maintained in fixed ratio; moreover, to this order, the relations between the $\delta\sigma_r$'s and the λ_r's are exactly those between the true stress-rates and the strain-rates (10). The total work of distortion, per unit *initial* volume, is then easily shown to be

$$\Sigma\left[\sigma_r + \tfrac{1}{2}(\delta\sigma_r - \sigma_r\,\delta\rho/\rho)\right]\lambda_r$$

to second order, where ρ is the density. In terms of components referred to the x_i axes this is

$$\left[\sigma_{ij} + \tfrac{1}{2}(\delta\sigma_{ij} - \sigma_{ij}\,\delta\rho/\rho\right]\lambda_{ij}$$

where, to second order,

$$\lambda_{ij} = e_{ij} - e_{ik}\,e_{jk}$$

and

$$e_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)$$

is the Lagrangian strain tensor. $\delta\sigma_{ij}$ denotes the change in the stress components along axes rotating with the element and initially coincident with the x_i axes.

With the help of the differential form of (3) we finally obtain the compact second-order formula

$$(s_{ij} + \tfrac{1}{2}\,\delta s_{ij})\frac{\partial u_j}{\partial x_i}$$

for the work of distortion per unit initial volume on the optimum path. In this it is to be understood that δs_{ij} and $\partial u_j/\partial x_i$ are related in the same way as $\dot{s}_{ij}$ and $\partial v_j/\partial x_i$, namely through (3) and (10) if the element is plastic, and through (6) if it is elastic. This expression for the work is evidently obtained also from a path in which the displacement field throughout the body is increased proportionately and continuously (possible if the material is compressible). It follows that we have certainly not *under*estimated the least work.

Equation (24) can now be replaced by

$$\int \delta s_{ij} \frac{\partial u_j}{\partial x_i} \, dV > 0 \qquad \text{or} \qquad \int \dot{s}_{ij} \frac{\partial v_j}{\partial x_i} \, dV > 0, \tag{25}$$

since the admissible fields can *now* be regarded indifferently as either displacement or velocity. Equivalently, by (3) and (11), the criterion is

$$2 \int E(\mathbf{v}) \, dV \equiv \int (\dot{\sigma} + \theta\sigma) \, \epsilon \, dV - \Sigma(\mathbf{v}) > 0 \tag{26}$$

for all fields $\mathbf{v}$ vanishing at the constraints. Now it was shown that the boundary-value problem set by rigid constraints on S_v and given nominal traction-rates on S_F has a unique solution when

$$\int \Delta(\dot{\sigma} + \theta\sigma) \, \Delta\epsilon \, dV - \Sigma(\Delta\mathbf{v}) > 0 \tag{27}$$

where $\Delta\mathbf{v}$ is the difference of any pair of fields vanishing at the constraints. $\Delta\dot{\sigma}$, $\Delta\epsilon$ are the differences of the corresponding stress-rates and strain-rates, but are not necessarily related by (10) in the plastic zone, on account of the non-linearity inherent in the different behaviour in loading and unloading. Consequently (26) and (27) are distinct criteria. Nevertheless, there is a definite connexion in that (26) is always satisfied when (27) is (but not vice versa); for (27) reduces to (26) when one field of a pair vanishes identically. A partly plastic state in which the boundary-value problem has a unique solution is therefore *certainly* stable; and even when the solution is not unique the state *may* still be stable. This is the general statement and proof of a conjecture by SHANLEY (1946, 1947).

The situation can be seen more clearly when the solid is elastically isotropic and h/μ has the same value everywhere in the plastic zone. With the notation of Section 5, and supposing that $\Sigma < H_\infty$, (26) becomes

$$\frac{h}{2\mu} > \alpha \qquad \text{where} \qquad \alpha = \text{Max}\left(\frac{\Sigma - H_0'}{H_\infty - \Sigma}\right) \qquad \text{in class } \mathbf{w}, \tag{28}$$

$$H_0'(\mathbf{w}) = H_\infty(\mathbf{w}) - \int 2\mu \, (\mathbf{n}\eta)^2 \, dV_p',$$

where V_p' is that part of the plastic zone where $\mathbf{n}\eta > 0$. This formula differs from (17) in that H_0' replaces H_0. Since $H_0' \geqslant H_0$ for any one field, $\beta \geqslant \alpha$. The critical values α and β can have either sign or vanish.

In the case $\beta > \alpha > 0$, if the rate of hardening is progressively lowered, keeping everything else fixed, there is reached first the value β when uniqueness is no longer certain and afterwards the value α when instability first occurs. Thus, for certain processes of distortion, a point of bifurcation can be reached at an earlier stage than a failure of stability (supposing, naturally, that the rate of hardening diminishes with continued strain). Moreover, at such a stable bifurcation the loading must change with further deformation in such a way that

$$\int \dot{\mathbf{F}}\mathbf{v} \, dS = 2 \int E(\mathbf{v}) \, dV > 0$$

where $\mathbf{v}$ here denotes the velocity in any of the possible actual modes.

The given state may be such that there are special values of $h/2\,\mu$ (eigenvalues) for which deformation can proceed under temporarily constant loading (i.e. loosely speaking, through adjacent positions of equilibrium). Since $\int E\,dV$ then vanishes for the actual velocity field, no eigenvalue can exceed the stability limit α corresponding to the given state. On the other hand, although the stability limit is itself marked by the vanishing of $\int E\,dV$ for *some* field, there seems no reason why this should necessarily be an actual field under dead loading nor α an eigenvalue. This does not appear to have been envisaged in previous work, which has been concerned almost exclusively with determining neighbouring positions of equilibrium. We may contrast this possibility sharply with the behaviour of an elastic solid (at least in the classical linear theory) where, according to Rayleigh's Principle, the stability functional remains positive as the varied parameter ranges right up to the first eigenvalue.

8. Comparison with Rigid–Plastic Theory

Consider the limiting form of the previous theorems when the elastic moduli are indefinitely increased, the current stresses and rate of hardening being held fixed. K is now taken to be positive-definite.

In the uniqueness criterion (16) the functional Σ is independent of the moduli while the asymptotic form of H is

$$H(\mathbf{w}) \doteq H_0(\mathbf{w}) + \int h \left(\frac{\mathbf{n}K\boldsymbol{\eta}}{\mathbf{n}K\mathbf{n}}\right)^2 dV_p$$

since $\mathbf{n}\,K\,\mathbf{n}$ never vanishes. H_0 is never negative, by the Schwarz inequality, and vanishes only for incompressible $\mathbf{w}$ fields such that $\boldsymbol{\eta} = 0$ in the elastic zone and $\boldsymbol{\eta} \propto \mathbf{n}$ in the plastic zone (the proportionality factor being *arbtirary in sign* and magnitude). For these special fields

$$H(\mathbf{w}) \doteq \int h\eta^2\,dV$$

while for all others H_0 and H can be made arbitrarily large and positive by increasing the moduli. Consequently the sufficient condition for uniqueness approaches the requirement that

$$\int h\eta^2\,dV - \Sigma(\mathbf{w}) > 0$$

for all *special* $\mathbf{w}$ fields vanishing on S_v. In particular, when the solid is plastically isotropic, the principal axes of $\mathbf{n}$, $\boldsymbol{\sigma}$ and the special $\boldsymbol{\eta}$ coincide and Σ can be written more concisely as

$$\int \sigma_{ik} \frac{\partial w_j}{\partial x_i} \frac{\partial w_k}{\partial x_j}\,dV.$$

These are exactly the results obtained for a rigid–plastic solid, when it is treated purely on its own merits and not as a limiting case (HILL 1957a*); that is, from

*The restriction there to convex yield surfaces is in no way essential to the proof and could be dropped forthwith (other considerations apart). A slight modification would arise since the stress in the deformable zone need not then be determined uniquely by the current tractions on S_F and velocities on S_v. Similarly, in an elastic-plastic solid the state of stress approached asymptotically under constant surface traction and velocity need not be unique anywhere (contrast HILL 1956a, p. 72) but would depend on the initial stresses.

the outset ϵ is put zero in the elastic zone while (8) is replaced by $h\,\epsilon = (\mathbf{n}\,\dot{\sigma})\,\mathbf{n}$ in the plastic zone. There are no inverse relations such as (10), for which reason the boundary value problem uniquely determines the strain-rate only and not the stress-rate, unless $S_v = 0$ (cf. HILL 1956b, p. 248).

The general connexion could not indeed be regarded in advance as automatic, if only on the ground that a property possessed by all members of a limiting sequence need not be a property of the limit itself. Not only that, but the setting of the boundary-value problem itself for a rigid–plastic solid is different in that, if the existing stresses are regarded as given, the velocity can not be prescribed arbitrarily on S_v. Conversely, if we choose to regard the velocity on S_v as arbitrarily prescribed, the associated stress in the deformable zone is decided, at least when the yield surface is convex (HILL 1951); if the pre-existing stress were different it would change instantaneously to become consistent with the imposed velocities (cf. HILL 1950, pp. 58-9).

Consider, next, the functional (19) in the extremum principle. As the elastic moduli are increased, the value of the functional is unbounded except for incompressible fields $\mathbf{v}$ such that $\epsilon = 0$ in the elastic zone and $\epsilon \propto \mathbf{n}$ in the plastic zone, with a *non-negative* factor of proportionality, when its limiting value is

$$\int h\epsilon^2\,dV - \Sigma(\mathbf{v}) - 2\int \dot{\mathbf{F}}\mathbf{v}\,dS_F.$$

Only such special $\mathbf{v}$ fields satisfying the velocity boundary-conditions need be considered in the principle. These fields exist and are just the set of virtual modes for a rigid–plastic solid under the given surface tractions and velocities, provided the latter are consistent with the given internal stress. Just such a principle was proved by HILL (1957b) for a rigid–plastic solid treated on its merits. But the minimum is only analytic when the actual solution involves no unloading (*op. cit.*, p. 304); this is therefore an example of a property *not* shared by the two solids.

Consider, finally, the stability criterion (26). In the limit, as before, we need only consider the set of virtual modes for a rigid–plastic solid under the given loads and constraints, the functional becoming

$$\int h\epsilon^2\,dV - \Sigma(\mathbf{v}).$$

This agrees with the results obtained for a rigid–plastic solid treated on its merits (HILL 1957d). However, for an elastic–plastic solid the uniqueness and stability functionals differ but the classes of admissible fields are identical, while for a rigid-plastic solid the functionals are the same but the admissible classes differ.

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